A Model for a Binary Burst-Noise Channel Based on a Finite Queue

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Abstract — We introduce a binary communication channel with memory whose noise is generated by a queue of length K. The queue is operated under two modes: uniform and non-uniform. The resulting noise process is shown to be a stationary and ergodic Markov source of order K. Analytic expressions for the noise stationary distribution, capacity and burst frequency of the uniform queue-based channel are presented. For the non-uniform queue-based channel, only numerical results are provided. Next, the capacity and burst frequency of the uniform and non-uniform queue-based channels are compared with those of the finite-memory Polya contagion channel and the Gilbert-Elliott channel.

1 Introduction

We introduce a binary communication channel with memory whose noise process is based on a finite queue of length K. More specifically, we consider the channel in two cases: a uniform queue-based mode where we experiment on the cells of the queue with equal probability, and a non-uniform queue-based mode where we experiment on the cells of the queue with different probabilities.

The statistical properties of the uniform queue-based channel are first investigated. The resulting channel noise is a stationary and ergodic Markov source of order K. Expressions for the noise stationary distribution, channel capacity and noise burst frequency are presented in terms of K. For the non-uniform queue-based channel, the noise is also stationary, ergodic and Markovian of order K. But we have no closed-form expression for the noise stationary distribution; hence, only numerical results are provided.

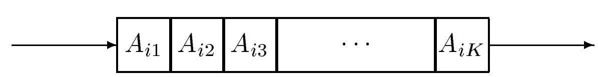
Next, the capacity and burst frequency of the uniform and non-uniform queue-based channels are compared with those of the finite-memory Polya contagion channel [1] and the Gilbert-Elliott channel [3]. It is shown (both analytically and numerically) that, surprisingly, the uniform queue-based channel and the finite-memory Polya contagion channel have an identical block transition probability when they have the same memory, bit error rate (BER) and correlation coefficient; hence, they have identical capacities and burst frequencies. When $q_1 \to 1$, the non-uniform case converges to the uniform case with memory $K = 1$. The non-uniform queue-based channel has lower burst frequencies than the uniform channel for low correlation coefficients, and it has higher burst frequencies for high correlation coefficients. Finally, the non-uniform queue-based channel has larger capacities than the uniform case when the queue probability $q_1 < \frac{1}{K}$, and it has smaller capacities than the uniform case when $q_1 > \frac{1}{K}$.

2 A Queue-Based Channel with Memory

In most real-world communications channels, noise distortion may produce errors in a bursty fashion; i.e., errors occur in clusters or bunches separated by fairly long error-free segments of data. This phenomenon is commonly know as "memory" [2]. In the quest to develop models that adequately represent real channel behavior and that are mathematically tractable, we present a binary channel with additive bursty noise based on a finite queue. It offers an interesting alternative to the Gilbert model and others.

Consider the binary channel given by $Y_i = X_i \oplus Z_i$, where X_i, Z_i, and Y_i are, respectively, the i^{th} input, noise, and output of the channel. We assume that the input and noise sources are independent of each other. Consider the following two parcels.

- **Parcel 1** is a queue of length K, that initially contains K balls.



  Let A_{ij} (i is a time index referring to the i^{th} experiment), $j = 1, 2, \cdots, K$, indicate the color of the ball in the corresponding cell of the queue at time i:

$$A_{ij} = \begin{cases} 1, & \text{if the } j^{th} \text{ cell contains a red ball,} \\ 0, & \text{if the } j^{th} \text{ cell contains a black ball.} \end{cases}$$

- **Parcel 2** is an urn that contains a very large number of balls where the proportion of black balls is $1 - p$ and the proportion of red balls is p, where $p \in (0, 1)$; usually $p \ll 1/2$.

Let the probability of selecting parcel 1 (the queue) be ε and the probability of selecting parcel 2 (the urn) be $1 - \varepsilon$;

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where $\varepsilon \in (0,1)$. The noise process $\{Z_i\}$ is generated by one of the following mechanisms.

Mechanism 1 Uniform queue-based channel with memory: *By flipping a biased coin (with $P(Head)=\varepsilon$), we select one of the 2 parcels (select the queue if Head and the urn if Tail). Then a pointer randomly points at a ball from the selected parcel, and identifies its color.*

Mechanism 2 Non-uniform queue-based channel with memory: *By flipping a biased coin (with $P(Head)=\varepsilon$), we select one of the 2 parcels (select the queue if Head and the urn if Tail). If parcel 1 (the queue) is selected, then a pointer points at the ball in cell 1 with probability q_1 and points at the ball in cell l with probability $q_l = (1-q_1)/(K-1)$, for $l = 2, 3, \cdots, K$, and identifies its color. If parcel 2 (the urn) is selected, a pointer randomly points at a ball, and identifies its color.*

If the selected ball is red, we introduce a red ball in cell 1 of the queue, pushing the last ball in cell K out. If the selected ball is black, we introduce a black ball in cell 1 of the queue, pushing the last ball in cell K out. The noise process $\{Z_i\}$ is then modeled as follows:

$$Z_i = \begin{cases} 1, & \text{if the } i^{th} \text{ experiment points at a red ball;} \\ 0, & \text{if the } i^{th} \text{ experiment points at a black ball.} \end{cases}$$

Definition 1 *For a given mechanism, define the state of the channel to be $\underline{S}_i \stackrel{\triangle}{=} (A_{i1},\ A_{i2}, \cdots, A_{iK})$, the binary $K-$tuple in the queue after the i^{th} experiment is completed. Note that, in terms of the noise process, the channel state at time i can be written as $\underline{S}_i = (Z_i, Z_{i-1}, \cdots, Z_{i-K+1})$, for $i \geq K$.*

2.1 Uniform Queue-Based Channel

Noise Properties: We now investigate the properties of the binary noise process $\{Z_n\}_{n=1}^{\infty}$. We first observe that $\{Z_n\}_{n=1}^{\infty}$ is a homogeneous Markov process of order K, since for $n \geq K+1$,

$$\begin{aligned} &\Pr[Z_n = 1 | Z_{n-1} = a_{n-1}, \cdots, Z_1 = a_1] \\ &= \varepsilon \frac{a_{n-1} + \cdots + a_{n-K}}{K} + (1-\varepsilon)p \\ &= \Pr[Z_n = 1 | Z_{n-1} = a_{n-1}, \cdots, Z_{n-K} = a_{n-K}], \end{aligned}$$

where $a_j \in \{0, 1\}$, $j = 1, \cdots, n$.

Throughout this work, we consider the case where the initial distribution of the Markov noise $\{Z_n\}$ is drawn according to its stationary distribution; hence the noise process $\{Z_n\}$ is stationary. $\{\underline{S}_n\}$ is a homogeneous Markov process with stationary (or initial) distribution [4]

$$\pi_i = \frac{1}{\prod_{m=1}^{K}(1-\varepsilon\frac{m}{K})} \prod_{j=0}^{K-1-\omega(i)} [\varepsilon \frac{j}{K} + (1-\varepsilon)(1-p)] \prod_{l=0}^{\omega(i)-1} [\varepsilon \frac{l}{K} + (1-\varepsilon)p],$$

for $i = 0, 1, 2, \cdots, 2^K - 1$, where $\omega(i)$ is the number of "ones" in the binary representation of the decimal integer i and $\prod_{i=0}^{a}(\cdot) \stackrel{\triangle}{=} 1$, if $a < 0$.

Block Transition Probability: For an input block $\underline{X} = [X_1, \cdots, X_n]$ and an output block $\underline{Y} = [Y_1, \cdots, Y_n]$, where n is the block length, the block transition probability of the resulting binary channel is as follows [4].

- For block length $n \leq K$,

$$\Pr(\underline{Y} = \underline{y} | \underline{X} = \underline{x}) = \frac{1}{\prod_{l=K-n+1}^{K}(1-\varepsilon\frac{l}{K})} \prod_{s=0}^{n-d-1} [\varepsilon \frac{s}{K} + (1-\varepsilon)(1-p)] \prod_{t=0}^{d-1} [\varepsilon \frac{t}{K} + (1-\varepsilon)p],$$

  where d is the number of "ones" in $\underline{x} \oplus \underline{y}$.

- For block length $n \geq K+1$,

$$\Pr(\underline{Y} = \underline{y} | \underline{X} = \underline{x}) = L \prod_{i=K+1}^{n} \left[\varepsilon \frac{\lambda_{i-1}}{K} + (1-\varepsilon)p\right]^{a_i} \left[\varepsilon \frac{K - \lambda_{i-1}}{K} + (1-\varepsilon)(1-p)\right]^{1-a_i},$$

  where $L = \prod_{j=0}^{K-1-\lambda_K} [\varepsilon \frac{j}{K} + (1-\varepsilon)(1-p)] \prod_{l=0}^{\lambda_K - 1} [\varepsilon \frac{l}{K} + (1-\varepsilon)p] / \prod_{t=1}^{K}(1-\varepsilon\frac{t}{K})$, $\prod_{i=0}^{a}(\cdot) \stackrel{\triangle}{=} 1$, if $a < 0$, $\lambda_{i-1} = a_{i-1} + \cdots + a_{i-K}$, and $a_i = x_i \oplus y_i$.

Capacity: The uniform queue-based channel with memory is a channel with stationary ergodic Markov additive noise of memory K and BER p. The channel capacity C_K is positive and non-decreasing in K and is given by

$$C_K = 1 - \sum_{i=0}^{K} \binom{K}{i} L_i h_b \left(\varepsilon \frac{i}{K} + (1-\varepsilon)p\right)$$

where $L_i = \prod_{j=0}^{K-1-i} [\varepsilon \frac{j}{K} + (1-\varepsilon)(1-p)]$, $\prod_{l=0}^{i-1} [\varepsilon \frac{l}{K} + (1-\varepsilon)p] / \prod_{m=1}^{K}(1-\varepsilon\frac{m}{K})$, and $\prod_{t=0}^{a}(\cdot) \stackrel{\triangle}{=} 1$ if $a < 0$, and $h_b(\cdot)$ is the binary entropy function.

Burst Frequency: Noise sequences of 1s between two 0s are called error bursts. The length of a burst is defined as one plus the total number of 1s in the noise sequence between two 0s. If B_n denotes the length of an error burst starting at time n and conditioned on $Z_n = 0$, then we obtain the following (cf. [4]).

- For $1 \leq l \leq K-1$, where $K > 1$,

$$\Pr[B_n = l] = \frac{1}{1-p} \cdot \frac{1}{\prod_{u=K-l}^{K}(1-\varepsilon\frac{u}{K})} \prod_{s=0}^{1} [\varepsilon \frac{s}{K} + (1-\varepsilon)(1-p)] \prod_{t=0}^{l-2} [\varepsilon \frac{t}{K} + (1-\varepsilon)p].$$

- For $l = K$,

$$\Pr[B_n = K] = \prod_{t=0}^{K-2} [\varepsilon \frac{t}{K} + [(1-\varepsilon)p] \frac{[(1-\varepsilon)(1-p)] \cdot [\varepsilon \frac{1}{K} + (1-\varepsilon)(1-p)]}{(1-p)\prod_{u=1}^{K}(1-\varepsilon\frac{u}{K})}.$$

- For $l \geq K+1$,

$$\Pr[B_n = l] = \frac{\prod_{t=0}^{K-2}[\varepsilon\frac{t}{K} + [(1-\varepsilon)p]}{(1-p)\prod_{u=1}^{K}(1-\varepsilon\frac{u}{K})} [(1-\varepsilon)(1-p)] \cdot [\varepsilon\frac{K-1}{K} + (1-\varepsilon)p] \cdot [\varepsilon + (1-\varepsilon)p]^{l-K-1} \cdot [(1-\varepsilon)(1-p)].$$

2.2 Non-Uniform Queue-Based Channel

For the non-uniform queue-based channel, the noise is also stationary, ergodic and Markovian of order K. But we have no analytical expression for the noise stationary distribution in terms of K; hence, only numerical results are given for specific values of K.

Capacity: We take $K = 3$ as an example.

$$C_3 = 1 - [-\sum_{i,j=0}^{7} \pi_i p_{ij} \log_2 p_{ij}],$$

where $[p_{ij}]$ is the noise transition probability matrix.

Burst Frequency: We take $K = 2$ as an example.

- For $l = 1$, $\Pr[B_n = l] = \frac{\pi_0}{1-p}$.
- For $l = 2$,

$$\Pr[B_n = l] = \frac{\pi_2}{1-p} \cdot [\varepsilon(1-q_1) + (1-\varepsilon)(1-p)].$$

- For $l \geq 3$,

$$\Pr[B_n = l] = \frac{\pi_2}{1-p} \cdot [\varepsilon q_1 + (1-\varepsilon)p] \cdot [\varepsilon + (1-\varepsilon)p]^{l-3} \cdot [(1-\varepsilon)(1-p)].$$

3 Comparisons with other Channels

We next compare the uniform queue-based channel with the Polya contagion [1] and Gilbert-Elliott [3] channels in terms of capacity and burst frequency. Similar comparisons are made for the non-uniform queue-based channel.

We first observe that it can be shown analytically [4] that the finite-memory contagion channel and the uniform queue-based channel are surprisingly identical; i.e., they have the same block transition probability for the same memory K, BER and noise correlation coefficient Cor. Therefore the two channels have identical capacities and burst frequencies under the above conditions.

In Figs. 1-6, capacity and burst frequency results are presented for the four channels under various channel conditions. For the Gilbert-Elliott channel the parameter p_G represents the channel BER when the channel is in a good state, while p_B denotes the BER under a bad channel state. Throughout these figures, we let $p_G = 2 \times 10^{-5}$ and $p_B = 0.92$. For the non-uniform queue-based channel, the cell probability $q_1 = 0.9$ was used.

We note that capacity increases as Cor increases (Figs. 1-2) and as BER decreases (Fig. 3), as expected. For the uniform queue-based and the contagion channels, capacity also increases with K (Figs. 1-2). When $Cor = 0.1$, the capacities of the uniform queue-based and contagion channels are always larger than that of the Gilbert-Elliott channel for any K (Fig. 1). But as Cor increases, the capacity of the Gilbert-Elliott channel grows faster. When $Cor = 0.9$, the uniform queue-based channel and the contagion channel have lower capacities than the Gilbert-Elliott channel for small Ks and have higher capacities for large Ks (Fig. 2).

It is clear from Fig. 3 and Fig. 4 that the three channels have almost equal capacities and burst frequencies when $K = 1$. This means that in these cases we can replace the Gilbert-Elliott channel with the (less complex) uniform queue-based channel (or the contagion channel) if our target is to achieve an error burst behavior and capacity that are close to those of the Gilbert-Elliott channel.

The non-uniform queue-based channel has lower burst frequencies than the uniform channel for low values of Cor (Fig. 5). But it has higher burst frequencies for high values of Cor and burst length ≥ 3 (Fig. 6). But the burst frequencies of the non-uniform channel decreases faster than those of the uniform channel; thus the former eventually has lower burst frequency when the burst length is big enough. We notice that the non-uniform channel has similar burst frequency as the Gilbert-Elliott channel. This is because the non-uniform channel was used with $q_1 = 0.9$, and as $q_1 \to 1$ the channel converges to the uniform case with $K = 1$ (see Fig. 4).

Finally, we observe (see [4]) that the non-uniform queue-based channel has larger capacities than the uniform case when the queue probability $q_1 < \frac{1}{K}$, and it has smaller capacities than the uniform case when $q_1 > \frac{1}{K}$.

References

[1] F. Alajaji and T. Fuja, "A communication channel modeled on contagion," *IEEE Trans. Inform. Theory*, Vol. 40, pp. 2035-2041, Nov. 1994.

[2] L. N. Kanal and A. R. K. Sastry, "Models for channels with memory and their applications to error control," *Proc. IEEE* , Vol. 66, pp. 724-744, July 1978.

[3] M. Mushkin and I. Bar-David, "Capacity and coding for the Gilbert-Elliott channel," *IEEE Trans. Inform. Theory*, Vol. 35, pp. 1277-1290, Nov. 1989.

[4] L. Zhong, "A binary channel with additive bursty noise based on a finite queue," *M.Sc. Project*, Department of Mathematics and Statistics, Queen's University, Nov. 2000. Available at: *http://markov.mast.queensu.ca/publications.html*.

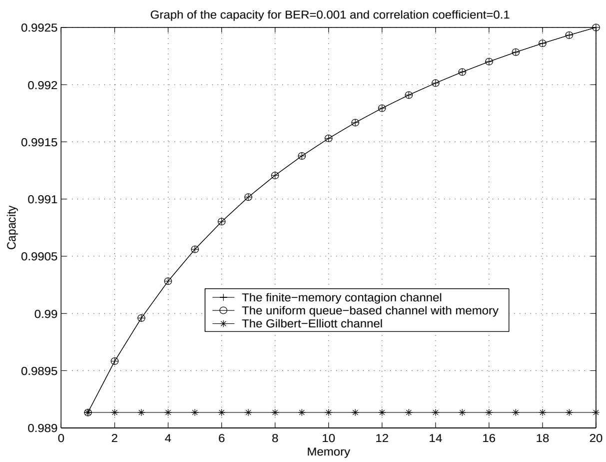

Figure 1: Capacity vs. K for BER=0.001 and Cor=0.1.

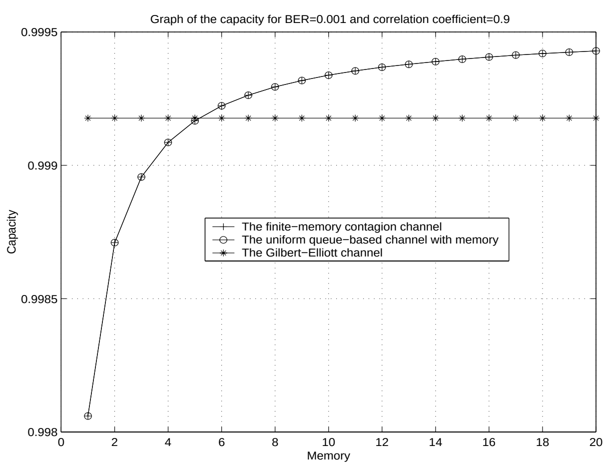

Figure 2: Capacity vs. K for BER=0.001 and Cor=0.9.

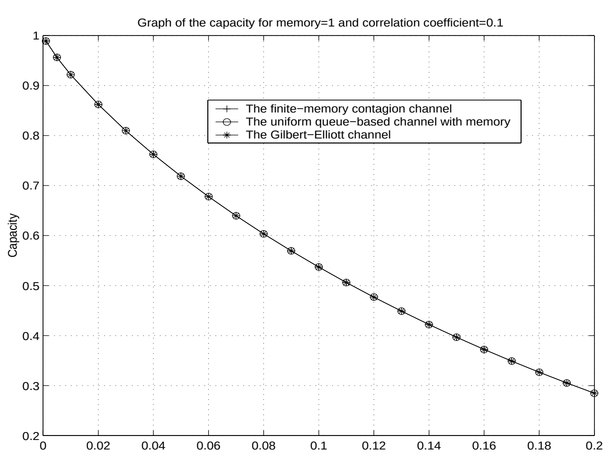

Figure 3: Capacity vs. BER for $K = 1$ and Cor=0.1.

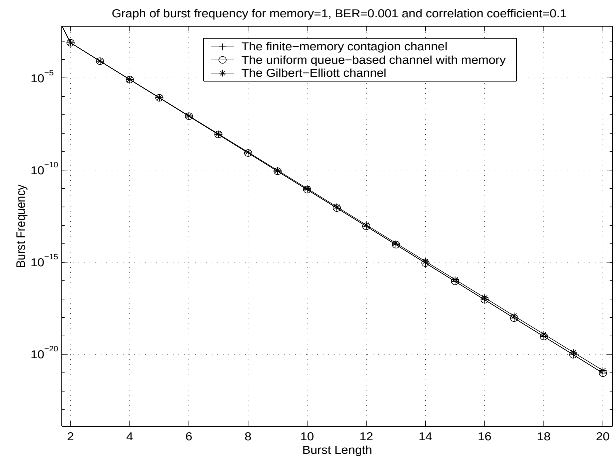

Figure 4: Burst frequency vs. burst length for $K = 1$, BER=0.001 and Cor=0.1.

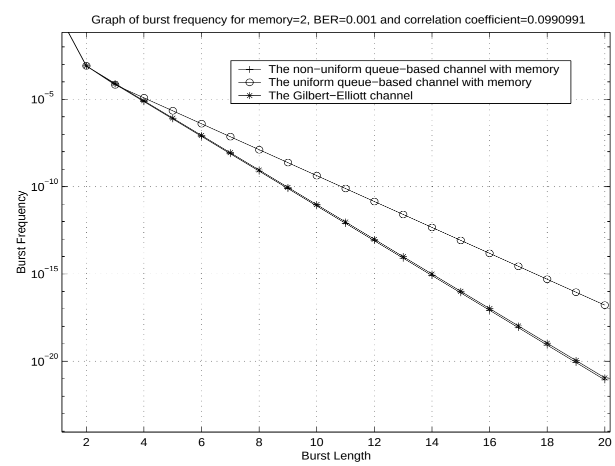

Figure 5: Burst frequency vs. burst length for $K = 2$, BER=0.001 and Cor=0.0990991.

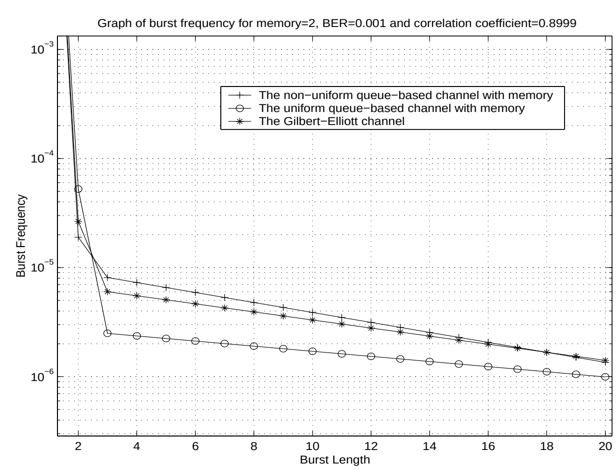

Figure 6: Burst frequency vs. burst length for $K = 2$, BER=0.001 and Cor=0.8999.